

A central limit theorem under sublinear expectations[☆]

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Abstract

In this paper we consider a sequence of random variables with mean uncertainty in a sublinear expectation space. Without the hypothesis of identical distributions, we show a new central limit theorem under the sublinear expectations.

Keywords: central limit theorem, sublinear expectation, G -normal distribution, mean uncertainty

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1. Introduction

Motivated by the coherent risk measures (cf. [1, 2]) and uncertain volatility models in finance (see, e.g. [11]), Peng [12, 14] recently introduced the notion of sublinear expectation which is not based on a classical probability space. Under the sublinear expectations, a random variable X in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ (for its definition, see Section 2 of this paper) is said to be of G -normal distribution with zero mean (cf. [14, 17]), if for each Y which is an independent copy of X , it holds that

$$aX + bY \stackrel{d}{=} \sqrt{a^2 + b^2}X, \quad \forall a, b \geq 0.$$

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Just as the classical normal distributions in probability theory, for a G -normal distributed random variable X , we have (cf. [14])

$$\hat{E}[\varphi(X)] = u(1, 0), \quad \forall \varphi \in C_{b,Lip}(R),$$

where $u(t, x)$ is the unique viscosity solution for the following heat equation

$$\begin{cases} \partial_t u - G(\partial_{xx}^2 u) = 0, \\ u|_{t=0} = \varphi, \end{cases}$$

where $G(\alpha) := \hat{E}[\frac{1}{2}\alpha X^2]$. In the theory of sublinear expectations, the above heat equation often plays a role of characteristic function in probability theory.

On the basis of G -normal distribution, G -Brownian motion can be defined, and the corresponding stochastic calculus with respect to the G -Brownian motions and the related Itô's formula can also be established (cf. [12, 14]). Since the importance of law of large numbers (LLN) and central limit theorem (CLT) in probability theory, Peng [13, 15] has shown the corresponding LLN and CLT under sublinear expectations, which indicate that G -normal distributions play the same important role in the theory of sublinear expectations as the normal distributions in the classical probability theory.

Due to the significance of sublinear expectations in finance and statistics, the theory of sublinear expectations has been attracting more and more attentions in both pure and applied mathematics (see, e.g. [6], [8], [18], [19] and [20]).

The purpose of this paper is to investigate one of the very important fundamental results in the theory of sublinear expectations—Central Limit Theorem. Until now, all the results on central limit theorems under sublinear expectations require that the sequence of random variables is independent and identically distributed. Analogous to the CLT in the probability theory, a natural question is whether one can weaken the hypothesis of identical distributions for the CLT under sublinear expectations?

In this paper, without the hypothesis of identical distributions, we prove a new central limit theorem within mean uncertainty under the sublinear expectations, which extends Peng's results. Precisely, for a sequence of random variables $\{(X_n, Y_n)\}_{n=1}^\infty$ in a sublinear expectation space, we only require the random variable (X_{n+1}, Y_{n+1}) is independent to $\{(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)\}$, $n = 1, 2, 3, \dots$.

2. Basic settings

Let Ω be a given set and \mathcal{H} a vector lattice of real functions defined on Ω , including 1, such that if $X_1, \dots, X_n \in \mathcal{H}$ then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{b,Lip}(R^n)$ where $C_{b,Lip}(R^n)$ denotes the space of bounded and Lipschitz continuous functions defined on R^n . \mathcal{H} is considered as a space of “random variables”. We denote by $\langle x, y \rangle$ the scalar product of $x, y \in R^n$ and by $|x| = \langle x, x \rangle^{1/2}$ the Euclidean norm of $x \in R^n$. Let $S(n)$ be the collection of $n \times n$ -symmetric matrices. $S(n)$ is obviously a Hilbert space with the scalar product $\langle P, Q \rangle = \text{tr}[PQ]$.

Now we give some related definitions about sublinear expectations (see [12-17] for details).

Definition 2.1. A sublinear expectation \hat{E} on \mathcal{H} is a functional $\hat{E} : \mathcal{H} \rightarrow R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) Monotonicity: If $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$.
- (b) Constant preserving: $\hat{E}[c] = c, \forall c \in R$.
- (c) Sub-additivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$.
- (d) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X], \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2. Let X_1 and X_2 be two n -dimensional random vectors defined in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$, respectively.

They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{b,Lip}(R^n).$$

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}[\cdot]$ if for each test function $\varphi \in C_{b,Lip}(R^m \times R^n)$ we have

$$\hat{E}[\varphi(X, Y)] = \hat{E}\left[\hat{E}[\varphi(x, Y)]_{x=X}\right].$$

The definition of independence means that any realization of X does not change the distributional uncertainty of Y . At the same time, the fact that Y is independent to X does not imply that X is independent to Y , an example can be seen in [14] or [15].

Definition 2.4. (*G*-normal distribution with mean uncertainty) A pair of d -dimensional random vectors (X, Y) in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called *G*-normal distributed if for each $a, b \geq 0$ we have

$$(aX + b\bar{X}, a^2Y + b^2\bar{Y}) \stackrel{d}{=} \left(\sqrt{a^2 + b^2} X, (a^2 + b^2) Y \right), \quad \forall a, b \geq 0,$$

where (\bar{X}, \bar{Y}) is an independent copy of (X, Y) .

The following propositions (cf. [12, 15]) play an important role in this paper.

Proposition 2.5. *Let $G : R^d \times S(d) \rightarrow R$ be a given sublinear functional continuous in $(0, 0)$ and satisfying the following properties: for any (p, A) and $(\bar{p}, \bar{A}) \in R^d \times S(d)$*

$$\begin{cases} G(p + \bar{p}, A + \bar{A}) \leq G(p, A) + G(\bar{p}, \bar{A}), \\ G(\lambda p, \lambda A) = \lambda G(p, A), \quad \forall \lambda \geq 0, \\ G(p, A) \geq G(p, \bar{A}), \quad \text{if } A \geq \bar{A}. \end{cases}$$

*Then there exists a pair of d -dimensional *G*-normal distributed random vectors (X, Y) in some sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ such that*

$$G(p, A) = \hat{E} \left[\frac{1}{2} \langle AX, X \rangle + \langle p, Y \rangle \right], \quad \forall (p, A) \in R^d \times S(d).$$

Proposition 2.6. *Let (X, Y) be a *G*-normal distributed random vector in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$. For each $\varphi \in C_{b,Lip}(R^d)$ we define a function*

$$v(t, x) := \hat{E} \left[\varphi \left(x + \sqrt{t}X + tY \right) \right], \quad \forall (t, x) \in [0, \infty) \times R^d.$$

Then v is the unique viscosity solution of the following parabolic partial differential equation (PDE)

$$\partial_t v - G(D_x v, D_x^2 v) = 0, \quad v|_{t=0} = \varphi,$$

where

$$G(p, A) = \hat{E} \left[\frac{1}{2} \langle AX, X \rangle + \langle p, Y \rangle \right], \quad \forall (p, A) \in R^d \times S(d).$$

Proposition 2.7. *Let X, Y be two random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, then for $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\hat{E} [|XY|] \leq \left(\hat{E} [|X|^p] \right)^{1/p} \cdot \left(\hat{E} [|Y|^q] \right)^{1/q},$$

In particular, for $1 \leq p \leq p'$, we have

$$\left(\hat{E} [|X|^p] \right)^{1/p} \leq \left(\hat{E} [|X|^{p'}] \right)^{1/p'}.$$

3. Main results

Now we give the main result in this paper—Central Limit Theorem. For the simplicity of notations, we first prove the 1-dimensional case of CLT.

Theorem 3.1. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, let $\{(X_i, Y_i)\}_{i=1}^\infty$ be a sequence of $R \times R$ -valued random variables and (ξ, ζ) be a pair of G -normal distributed random variables. We assume that*

- (i) (X_{i+1}, Y_{i+1}) is independent to $\{(X_1, Y_1), \dots, (X_i, Y_i)\}$, for $i = 1, 2, \dots$;
- (ii) $\hat{E} [X_i] = \hat{E} [-X_i] = 0$, $\hat{E} [|X_i|^3] \leq M$, $\hat{E} [|Y_i|^3] \leq M$, where M is a positive constant;
- (iii) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \hat{E} [|X_i^2 - \xi^2|^2] = 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \hat{E} [|Y_i - \zeta|^2] = 0$;
- (iv) there exists $\beta > 0$, such that $\hat{E} [a\xi^2] - \hat{E} [\bar{a}\xi^2] \geq \beta(a - \bar{a})$, for any $a, \bar{a} \in R$ with $a \geq \bar{a}$.

Then the sequence $\left\{ \frac{S_n}{\sqrt{n}} + \frac{T_n}{n} \right\}_{n=1}^\infty$, where $S_n = X_1 + \dots + X_n$, $T_n = Y_1 + \dots + Y_n$, converges in law to $\xi + \zeta$:

$$\lim_{n \rightarrow \infty} \hat{E} \left[\varphi \left(\frac{S_n}{\sqrt{n}} + \frac{T_n}{n} \right) \right] = \hat{E} [\varphi(\xi + \zeta)], \quad \forall \varphi \in C_{b,Lip}(R),$$

where the sublinear function $G : R \times R \rightarrow R$ is defined by

$$G(p, a) := \hat{E} \left[p\zeta + \frac{1}{2}a\xi^2 \right].$$

Proof. For any $\varphi \in C_{b,Lip}(R)$, and a small but fixed $h > 0$, let V be the unique viscosity solution of

$$\partial_t V + G(\partial_x V, \partial_{xx}^2 V) = 0, \quad (t, x) \in [0, 1+h] \times R, \quad V|_{t=1+h} = \varphi. \quad (1)$$

Since (ξ, ζ) is of G -normal distribution, from Proposition 6, we have

$$V(t, x) = \hat{E} \left[\varphi \left(x + \sqrt{1+h-t} \xi + (1+h-t) \zeta \right) \right].$$

Particularly,

$$V(h, 0) = \hat{E}[\varphi(\xi + \zeta)], \quad V(1+h, x) = \varphi(x). \quad (2)$$

Since (1) is a uniformly parabolic PDE and G is a convex function, thus, by the interior regularity of V (see Krylov [10], Theorem 6.2.3), we have

$$|V|_{C^{1+\alpha/2, 2+\alpha}([0,1] \times R)} < \infty,$$

for some $\alpha \in (0, 1)$.

We set $\delta = \frac{1}{n}$, $S_0 = T_0 = 0$ and $\bar{S}_i = \sqrt{\delta} S_i + \delta T_i$, then

$$\begin{aligned} & V(1, \sqrt{\delta} S_n + \delta T_n) - V(0, 0) \\ &= V(1, \bar{S}_n) - V(0, 0) \\ &= \sum_{i=0}^{n-1} \{V((i+1)\delta, \bar{S}_{i+1}) - V(i\delta, \bar{S}_i)\} \\ &= \sum_{i=0}^{n-1} \{[V((i+1)\delta, \bar{S}_{i+1}) - V(i\delta, \bar{S}_{i+1})] + [V(i\delta, \bar{S}_{i+1}) - V(i\delta, \bar{S}_i)]\} \\ &= \sum_{i=0}^{n-1} \{I_\delta^i + J_\delta^i\}, \end{aligned}$$

with, by Taylor's expansion,

$$\begin{aligned} J_\delta^i &= \partial_t V(i\delta, \bar{S}_i) \delta + \frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) X_{i+1}^2 \delta + \partial_x V(i\delta, \bar{S}_i) (X_{i+1} \sqrt{\delta} + Y_{i+1} \delta), \\ I_\delta^i &= \int_0^1 (\partial_t V(i\delta + \beta\delta, \bar{S}_{i+1}) - \partial_t V(i\delta, \bar{S}_{i+1})) d\beta\delta + (\partial_t V(i\delta, \bar{S}_{i+1}) - \partial_t V(i\delta, \bar{S}_i)) \delta \\ &\quad + \int_0^1 \int_0^1 \left[\partial_{xx}^2 V(i\delta, \bar{S}_i + \gamma\beta (X_{i+1} \sqrt{\delta} + Y_{i+1} \delta)) - \partial_{xx}^2 V(i\delta, \bar{S}_i) \right] \beta d\gamma d\beta \\ &\quad \times (X_{i+1} \sqrt{\delta} + Y_{i+1} \delta)^2 + \frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) (Y_{i+1}^2 \delta^2 + 2X_{i+1} Y_{i+1} \delta^{3/2}). \end{aligned}$$

Thus

$$\begin{aligned} \hat{E} \left[\sum_{i=0}^{n-1} J_{\delta}^i \right] - \hat{E} \left[- \sum_{i=0}^{n-1} I_{\delta}^i \right] &\leq \hat{E} [V(1, \bar{S}_n)] - V(0, 0) \\ &\leq \hat{E} \left[\sum_{i=0}^{n-1} J_{\delta}^i \right] + \hat{E} \left[\sum_{i=0}^{n-1} I_{\delta}^i \right]. \end{aligned} \quad (3)$$

For the 3rd term of J_{δ}^i , by (i) and (ii) we have

$$\hat{E} \left[\partial_x V(i\delta, \bar{S}_i) X_{i+1} \sqrt{\delta} \right] = \hat{E} \left[-\partial_x V(i\delta, \bar{S}_i) X_{i+1} \sqrt{\delta} \right] = 0.$$

We then combine the above equality with (1) as well as the independence of (X_{i+1}, Y_{i+1}) to $\{(X_1, Y_1), \dots, (X_i, Y_i)\}$, it follows that

$$\begin{aligned} \hat{E} [J_{\delta}^i] &= \hat{E} \left[\partial_t V(i\delta, \bar{S}_i) \delta + \frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) X_{i+1}^2 \delta + \partial_x V(i\delta, \bar{S}_i) Y_{i+1} \delta \right] \\ &= \hat{E} \left[\partial_t V(i\delta, \bar{S}_i) \delta + \hat{E} \left[\frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) X_{i+1}^2 \delta + \partial_x V(i\delta, \bar{S}_i) Y_{i+1} \delta \right] \right] \\ &\leq \delta \hat{E} \left[\partial_t V(i\delta, \bar{S}_i) + \hat{E} \left[\frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) \xi^2 + \partial_x V(i\delta, \bar{S}_i) \zeta \right] \right] \\ &\quad + \delta \hat{E} \left[\frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) (X_{i+1}^2 - \xi^2) + \partial_x V(i\delta, \bar{S}_i) (Y_{i+1} - \zeta) \right] \\ &= \delta \hat{E} \left[\frac{1}{2} \partial_{xx}^2 V(i\delta, \bar{S}_i) (X_{i+1}^2 - \xi^2) + \partial_x V(i\delta, \bar{S}_i) (Y_{i+1} - \zeta) \right] \\ &\leq \frac{\delta}{2} \hat{E} [\partial_{xx}^2 V(i\delta, \bar{S}_i) (X_{i+1}^2 - \xi^2)] + \delta \hat{E} [\partial_x V(i\delta, \bar{S}_i) (Y_{i+1} - \zeta)] \\ &\leq \frac{\delta}{2} \left(\hat{E} [|\partial_{xx}^2 V(i\delta, \bar{S}_i)|^2] \right)^{1/2} \cdot \left(\hat{E} [X_{i+1}^2 - \xi^2]^2 \right)^{1/2} \\ &\quad + \delta \left(\hat{E} [|\partial_x V(i\delta, \bar{S}_i)|^2] \right)^{1/2} \cdot \left(\hat{E} [Y_{i+1} - \zeta]^2 \right)^{1/2}. \end{aligned}$$

But since $\partial_{xx}^2 V$ is uniformly $\frac{\alpha}{2}$ -Hölder continuous in t and α -Hölder continuous in x on $[0, 1] \times R$, it follows that

$$|\partial_{xx}^2 V(i\delta, \bar{S}_i) - \partial_{xx}^2 V(0, 0)| \leq C \left(|\bar{S}_i|^{\alpha} + |i\delta|^{\frac{\alpha}{2}} \right),$$

where C is some positive constant. We note that

$$\begin{aligned}
\hat{E} \left[|\overline{S}_i|^{2\alpha} \right] &\leq \hat{E} \left[|\overline{S}_i|^{2\alpha} \vee 1 \right] \\
&= \hat{E} \left[(|\overline{S}_i| \vee 1)^{2\alpha} \right] \\
&\leq \hat{E} \left[(|\overline{S}_i| \vee 1)^2 \right] \\
&\leq \hat{E} \left[(|\overline{S}_i| + 1)^2 \right] \\
&\leq 2\hat{E} \left[|\overline{S}_i|^2 \right] + 2 \\
&\leq \frac{4}{n} \hat{E} \left[\left(\sum_{j=1}^i X_j \right)^2 \right] + \frac{4}{n^2} \hat{E} \left[\left(\sum_{j=1}^i Y_j \right)^2 \right] + 2 \\
&\leq \frac{4}{n} \sum_{j=1}^n \hat{E} [X_j^2] + \frac{4}{n} \sum_{j=1}^n \hat{E} [Y_j^2] + 2,
\end{aligned}$$

at the same time,

$$\begin{aligned}
\frac{1}{n} \sum_{j=1}^n \hat{E} [Y_j^2] &= \frac{1}{n} \sum_{j=1}^n \hat{E} [(Y_j - \zeta + \zeta)^2] \\
&\leq \frac{2}{n} \sum_{j=1}^n \hat{E} [(Y_j - \zeta)^2] + 2\hat{E} [\zeta^2].
\end{aligned}$$

From (iii), it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\hat{E} [|X_i^2 - \xi^2|^2] \right)^{1/2} = 0, \tag{4}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\hat{E} [|Y_i - \zeta|^2] \right)^{1/2} = 0, \tag{5}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \hat{E} [X_j^2] = \hat{E} [\xi^2].$$

Then there exists a constant $C_1 > 0$ such that

$$\left(\hat{E} [|\partial_{xx}^2 V(i\delta, \overline{S}_i)|^2] \right)^{1/2} \leq 2C_1.$$

Since $\varphi \in C_{b,Lip}(R)$, there exists a constant $C_2 > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq C_2 |x - y|, \quad \forall x, y \in R.$$

Then $\forall t \in [0, 1]$, $\forall x, y \in R$, we have

$$\begin{aligned} & |V(t, x) - V(t, y)| \\ &= \left| \hat{E} \left[\varphi \left(x + \sqrt{1+h-t}\xi + (1+h-t)\zeta \right) \right] - \hat{E} \left[\varphi \left(y + \sqrt{1+h-t}\xi + (1+h-t)\zeta \right) \right] \right| \\ &\leq \hat{E} \left[\left| \varphi \left(x + \sqrt{1+h-t}\xi + (1+h-t)\zeta \right) - \varphi \left(y + \sqrt{1+h-t}\xi + (1+h-t)\zeta \right) \right| \right] \\ &\leq C_2 |x - y|. \end{aligned}$$

From the above inequality, we obtain

$$|\partial_x V(t, x)| \leq C_2, \quad \forall (t, x) \in [0, 1] \times R.$$

Thus

$$\hat{E} [J_\delta^i] \leq \frac{C_1}{n} \left(\hat{E} [|X_{i+1}^2 - \xi^2|^2] \right)^{1/2} + \frac{C_2}{n} \left(\hat{E} [|Y_{i+1} - \zeta|^2] \right)^{1/2},$$

it follows that

$$\begin{aligned} \hat{E} \left[\sum_{i=1}^{n-1} J_\delta^i \right] &\leq \sum_{i=1}^{n-1} \hat{E} [J_\delta^i] \\ &\leq \frac{C_1}{n} \sum_{i=1}^{n-1} \left(\hat{E} [|X_{i+1}^2 - \xi^2|^2] \right)^{1/2} + \frac{C_2}{n} \sum_{i=1}^{n-1} \left(\hat{E} [|Y_{i+1} - \zeta|^2] \right)^{1/2}, \end{aligned}$$

then by (4) and (5), we have

$$\lim_{n \rightarrow \infty} \hat{E} \left[\sum_{i=0}^{n-1} J_\delta^i \right] \leq 0.$$

Similarly, we also have

$$\begin{aligned} \hat{E} [J_\delta^i] &\geq -\frac{\delta}{2} \left(\hat{E} [|\partial_{xx}^2 V(i\delta, \overline{S}_i)|^2] \right)^{1/2} \left(\hat{E} [|X_{i+1}^2 - \xi^2|^2] \right)^{1/2} \\ &\quad -\delta \left(\hat{E} [|\partial_x V(i\delta, \overline{S}_i)|^2] \right)^{1/2} \left(\hat{E} [|Y_{i+1} - \zeta|^2] \right)^{1/2}, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \hat{E} \left[\sum_{i=0}^{n-1} J_{\delta}^i \right] \geq 0.$$

Thus

$$\lim_{n \rightarrow \infty} \hat{E} \left[\sum_{i=0}^{n-1} J_{\delta}^i \right] = 0. \quad (6)$$

For I_{δ}^i , since both $\partial_t V$ and $\partial_{xx}^2 V$ are uniformly $\frac{\alpha}{2}$ -Hölder continuous in t and α -Hölder continuous in x on $[0, 1] \times R$, we have

$$\begin{aligned} |I_{\delta}^i| &\leq C_3 \delta^{1+\alpha/2} \left(1 + \left| X_{i+1} + \sqrt{\delta} Y_{i+1} \right|^{\alpha} + \left| X_{i+1} + \sqrt{\delta} Y_{i+1} \right|^{2+\alpha} \right) \\ &\quad + \frac{1}{2} \left| \partial_{xx}^2 V(i\delta, \overline{S}_i) (Y_{i+1}^2 \delta^2 + 2X_{i+1} Y_{i+1} \delta^{3/2}) \right|, \end{aligned}$$

where C_3 is some positive constant. From (i), it follows that

$$\begin{aligned} \hat{E} \left[\left| \partial_{xx}^2 V(i\delta, \overline{S}_i) X_{i+1} Y_{i+1} \delta^{3/2} \right| \right] &= \frac{1}{n^{3/2}} \hat{E} \left[\left| \partial_{xx}^2 V(i\delta, \overline{S}_i) \right| \right] \hat{E} \left[|X_{i+1} Y_{i+1}| \right] \\ &\leq \frac{2C_1}{n^{3/2}} \left(\hat{E} \left[|Y_{i+1}|^3 \right] \right)^{1/3} \left(\hat{E} \left[|X_{i+1}|^{3/2} \right] \right)^{2/3} \\ &\leq \frac{2C_1}{n^{3/2}} M^{1/3} \left(\hat{E} \left[|X_{i+1}|^3 \right] \right)^{1/3} \\ &\leq \frac{2C_1}{n^{3/2}} M^{2/3}, \end{aligned}$$

and

$$\hat{E} \left[\frac{1}{2} \left| \partial_{xx}^2 V(i\delta, \overline{S}_i) Y_{i+1}^2 \delta^2 \right| \right] \leq \frac{C_1}{n^2} M^{2/3}.$$

At the same time, we can claim that

$$\hat{E} \left[\left| X_{i+1} + \sqrt{\delta} Y_{i+1} \right|^{2+\alpha} \right] \leq \hat{E} \left[\left| X_{i+1} + \sqrt{\delta} Y_{i+1} \right|^3 \right]^{\frac{2+\alpha}{3}} \leq (8M)^{\frac{2+\alpha}{3}}.$$

Therefore it follows that

$$\lim_{n \rightarrow \infty} \hat{E} \left[\sum_{i=0}^{n-1} I_{\delta}^i \right] \leq \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \hat{E} \left[|I_{\delta}^i| \right] = 0.$$

It also holds that

$$\lim_{n \rightarrow \infty} -\hat{E} \left[-\sum_{i=0}^{n-1} I_{\delta}^i \right] \geq \lim_{n \rightarrow \infty} -\sum_{i=0}^{n-1} \hat{E} [|I_{\delta}^i|] = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \hat{E} \left[\sum_{i=0}^{n-1} I_{\delta}^i \right] = \lim_{n \rightarrow \infty} -\hat{E} \left[-\sum_{i=0}^{n-1} I_{\delta}^i \right] = 0. \quad (7)$$

Therefore from (3), (6) and (7), we have

$$\lim_{n \rightarrow \infty} \hat{E} \left[V \left(1, \sqrt{\delta} S_n + \delta T_n \right) \right] = V(0, 0).$$

On the other hand, for each $t, t' \in [0, 1+h]$ and $x \in R$, we have

$$\left| V(t, x) - V(t', x) \right| \leq C \left(\sqrt{|t - t'|} + |t - t'| \right).$$

Thus $|V(h, 0) - V(0, 0)| \leq C(\sqrt{h} + h)$ and

$$\left| \hat{E} [V(1, \overline{S}_n)] - \hat{E} [\varphi(\overline{S}_n)] \right| = \left| \hat{E} [V(1, \overline{S}_n)] - \hat{E} [V(1+h, \overline{S}_n)] \right| \leq C(\sqrt{h} + h).$$

Then we can claim that

$$\overline{\lim}_{n \rightarrow \infty} \left| \hat{E} \left[\varphi \left(\sqrt{\delta} S_n + \delta T_n \right) \right] - \hat{E} [\varphi(\xi + \zeta)] \right| \leq 2C(\sqrt{h} + h).$$

Since h can be arbitrarily small, we obtain that

$$\lim_{n \rightarrow \infty} \hat{E} \left[\varphi \left(\sqrt{\delta} S_n + \delta T_n \right) \right] = \hat{E} [\varphi(\xi + \zeta)].$$

The proof is completed.

It is not difficult to obtain the following statement.

Remark 3.2. In addition of the assumptions of Theorem 3.1, if there is a sequence of $R \times R$ -valued random variables $\{(\overline{X}_i, \overline{Y}_i)\}_{i=1}^{\infty}$ in another sublinear expectation space $(\Omega_1, \mathcal{H}_1, \tilde{E})$, such that $(\overline{X}_{i+1}, \overline{Y}_{i+1})$ is independent to $\{(\overline{X}_1, \overline{Y}_1), \dots, (\overline{X}_i, \overline{Y}_i)\}$ and $(\overline{X}_i, \overline{Y}_i) \stackrel{d}{=} (X_i, Y_i)$, for $i = 1, 2, \dots$, then we also have

$$\lim_{n \rightarrow \infty} \tilde{E} \left[\varphi \left(\frac{\sum_{i=1}^n \overline{X}_i}{\sqrt{n}} + \frac{\sum_{i=1}^n \overline{Y}_i}{n} \right) \right] = \hat{E} [\varphi(\xi + \zeta)], \quad \forall \varphi \in C_{b,Lip}(R).$$

By the same arguments, we can claim the multi-dimensional case of CLT.

Theorem 3.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, let $\{(X_i, Y_i)\}_{i=1}^\infty$ be a sequence of $R^d \times R^d$ -valued random vectors and (ξ, ζ) be a pair of G -normal distributed d -dimensional random vectors. We assume that*

- (i) (X_{i+1}, Y_{i+1}) is independent to $\{(X_1, Y_1), \dots, (X_i, Y_i)\}$, for $i = 1, 2, \dots$;
- (ii) $\hat{E}[X_i] = \hat{E}[-X_i] = 0$ and $\hat{E}[|X_i|^3] \leq M$, $\hat{E}[|Y_i|^3] \leq M$, where M is a positive constant;
- (iii) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \hat{E}[|X_i X_i^T - \xi \xi^T|^2] = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \hat{E}[|Y_i - \zeta|^2] = 0$;
- (iv) there exists $\beta > 0$, such that $\hat{E}[\langle A\xi, \xi \rangle] - \hat{E}[\langle \bar{A}\xi, \xi \rangle] \geq \beta \text{tr}[A - \bar{A}]$, for any $A, \bar{A} \in S(d)$ with $A \geq \bar{A}$.

Then the sequence $\left\{ \frac{S_n}{\sqrt{n}} + \frac{T_n}{n} \right\}_{n=1}^\infty$, where $S_n = X_1 + \dots + X_n$, $T_n = Y_1 + \dots + Y_n$, converges in law to $\xi + \zeta$:

$$\lim_{n \rightarrow \infty} \hat{E} \left[\varphi \left(\frac{S_n}{\sqrt{n}} + \frac{T_n}{n} \right) \right] = \hat{E}[\varphi(\xi + \zeta)], \quad \forall \varphi \in C_{b,Lip}(R^d),$$

where the sublinear functional $G : R^d \times S(d) \rightarrow R$ is defined by

$$G(p, A) := \hat{E} \left[\langle p, \zeta \rangle + \frac{1}{2} \langle A\xi, \xi \rangle \right].$$

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Abstract

In this paper we consider a sequence of random variables with mean uncertainty in a sublinear expectation space. Without the hypothesis of identical distributions, we show a new central limit theorem under the sublinear expectations.

Keywords: *Keywords:* central limit theorem; sublinear expectation; G -normal distribution; mean uncertainty

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1.

Abstract

Keywords:

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